

On 1-loop diagrams in AdS space and the random disorder problem

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We study the complex scalar loop corrections to the boundary-boundary gauge two point function in pure AdS space in Poincare coordinates, in the presence of a boundary quadratic perturbation to the scalar. These perturbations correspond to double trace perturbations in the dual CFT and modify the boundary conditions of the bulk scalars in AdS. We find that, in addition to the usual UV divergences, the 1-loop calculation suffers from a divergence originating in the limit as the loop vertices approach the AdS horizon. We show that this type of divergence is independent of the boundary coupling, and making use of which we extract the finite relative variation of the imaginary part of the loop via Cutkosky rules as the boundary perturbation varies. Applying our methods to compute the effects of a time-dependent impurity to the conductivities using the replica trick in AdS/CFT, we find that generally an IR-relevant disorder reduces the conductivity and that in the extreme low frequency limit the correction due to the impurities overwhelms the planar CFT result even though it is supposedly $1/N^2$ suppressed. Comments on the effect of time-independent impurity in such a system are presented.

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I. INTRODUCTION

AdS/CFT correspondence[1–3] has proven itself an extremely powerful tool in extending our understanding of strongly coupled quantum field theories, stretching its influence into many different realms of physics. Particularly, there has been a surge of interest in applying these techniques in condensed matter systems in recent days (see for example a recent review [4] and references therein). Thus far, studies of AdS/CFT have predominantly focused on extracting the leading large N physics in the CFT via (semi)-classical supergravity computations in AdS. However, if one attempts to make contact with more realistic systems, N should be finite and there exist many circumstances in which $1/N$ correction is important. These corrections correspond to non-planar diagrams in the CFT, and quantum loop corrections in the AdS bulk, and in particular 1-loop diagrams in AdS give rise to $1/N^2$ suppressed corrections. A number of physical phenomena are known to show up only if one includes loop corrections in the bulk, such as the de Haas - van Alphen quantum oscillations [5, 6] in strongly coupled charged systems, hydrodynamic long-time tails of a fluid [7], and the holographic manifestation of the Mermin-Wagner theorem [8]. More recently it is also shown in [9] circumstances where the loop corrections can compete with the tree contribution in certain non-fermi-liquids. Consequently, there has been growing interest in the community in understanding quantum loops in the supergravity theory in AdS. Calculating loops in AdS space could entail extra complexities, such as additional divergences, due to the non-trivial properties of the geometry. It certainly is of importance to investigate this problem closely, clarifying possible obstacles and extracting physical implications.

As one of the original motivations of the current study, it has been proposed in [10] that one may incorporate the replica trick that is commonly adopted in the context of condensed matter systems to capture the effect of random disorder, through AdS/CFT correspondence. There, one introduces replicas of the AdS spaces and the bulk fields. The coupling to the random disorder appears on the AdS side as boundary terms that couple to the replicated bulk fields, thus effectively changing the boundary conditions those fields satisfy. These kind of boundary perturbations have been studied previously for example in [11–17], and they are shown to correspond to multiple trace, and in case of a quadratic boundary term double trace perturbations, in the dual CFT theory. It is also shown there that the effect of these boundary perturbations of a scalar field for example, are only mediated to other fields, such as the graviton or the photon, beginning at 1-loop order. This implies that the effect of disorder on transport coefficients such as conductivities only shows up in loops [18].

Motivated by these interesting studies, we take a phenomenological approach and consider charged scalar fields coupled to $U(1)$ gauge fields in pure AdS space in a general Einstein-Maxwell theory in $d + 1$ dimensions, and study particularly the scalar 1-loop Witten diagrams correcting the photon boundary-boundary 2-point function. We then read-off the conductivity of the dual theory via the usual AdS/CFT dictionary. These scalars are allowed to satisfy mixed boundary conditions, corresponding to the introduction of double trace boundary perturbations as mentioned above. It is found that the computation done in Poincare coordinates in Lorentzian signature suffers from a divergence that arises as both vertices in the loop approach the horizon (or the IR limit) where the geodesic distance between them vanishes. It is surprising, however, that such a divergence persists even within the imaginary part of the loop. More specifically, it manifests itself as a singularity in the integral over loop-momenta in the collinear limit[19], that cannot be cured even though the phase-space volume approaches zero there. These divergences, however, turn out to be independent of the strength of the boundary perturbation. We demonstrate this property by studying the asymptotics of the propagators in the near horizon limit and extract the precise forms of these singularities, even though the full analytic result of the loop integral is not generally expressible in terms of elementary functions whose properties are otherwise obscure. We obtain in this way both the coefficients and the powers of each singular terms analytically and find that they have simple dependence on the number of bulk dimensions.

Given the universal nature of these divergences, we extract finite results by considering the differences of diagrams evaluated at different boundary couplings. We managed to compute the relative conductivities for a wide range of boundary couplings f and find that they interpolate smoothly between the two conformal limits $f \rightarrow 0$ and $f \rightarrow \infty$. We then return to the study of random disorder via the replica trick, and compute the conductivities under the influence of random impurities. At $d = 2 + 1$, we give an example where the computation can be done exactly. We find that the presence of the impurities generally reduce the conductivity. Moreover in the low frequency limit the correction overwhelms the planar CFT result. This implies that a re-summation in the deep IR is probably necessary, but also suggests a possible resolution to the puzzle of how $1/N$ suppressed corrections could actually drastically change the IR behavior of the transport coefficients. A more complete analysis of these re-summation is however not pursued in the present paper.

The organization of the paper is as follows. In section II we present the form of scalar bulk-to-bulk propagators satisfying general mixed boundary condition at the AdS boundary. In section III we briefly review the vector bulk-to-boundary propagator in momentum space and the scalar-vector vertices that are relevant for the loop diagrams. We also introduce Cutkosky rules in AdS space and compute the imaginary part of the scalar 1-loop correction to the photon boundary-boundary 2-point function. We did not include similar contribution of other fields such as gravitons

because we are mainly interested in the leading dependence on the coupling of the boundary perturbation. We will show that a divergence arises but is independent of the boundary perturbation, and that the remaining finite part interpolates smoothly between the two conformal limits. We also make some comments on these divergences more generally in other loops in AdS space. In section IV we apply our method in the context of random disorder. We conclude our results in section V. Further details of the computations are relegated to the appendices.

II. SCALAR PROPAGATORS OF MIXED BOUNDARY CONDITIONS

Consider AdS space in $d + 1$ dimensions with the metric

$$ds^2 = \frac{1}{z^2}(-dt^2 + dz^2 + \sum_i^{d-1} dx_i^2), \quad (1)$$

the Green's function of scalar fields in AdS space of mass m satisfy the inhomogenous Klein-Gordon equation sourced by delta-functions:

$$\left(\frac{1}{\sqrt{g}} (\partial_{z_1} g^{zz} \sqrt{g} \partial_{z_1}) + g^{\mu\nu} \partial_\mu \partial_\nu - m^2 \right) G_{\Delta_-}(z_1, z_2, x_1, y_1) = \frac{1}{\sqrt{g}} \delta^d(x_1 - y_1) \delta(z_1 - z_2). \quad (2)$$

In general these scalar fields behave in the boundary limit $z \rightarrow 0$ as

$$\phi(z) \sim \alpha z^{\Delta_+} + \beta z^{\Delta_-}, \quad \Delta_{\pm} = \frac{d}{2} \pm \nu, \quad (3)$$

where we have defined

$$\nu = \sqrt{m^2 + \frac{d^2}{4}}. \quad (4)$$

When both Δ_{\pm} are greater than zero and thus the corresponding wave-function normalizable, one can insert boundary terms of the form

$$\delta S_{\partial\mathcal{M}} = \int_{\partial\mathcal{M}} f \beta^2, \quad (5)$$

so that the variation of the action only vanishes on the boundary if the scalar field satisfies the following boundary condition

$$\alpha = f \beta. \quad (6)$$

For finite f the dual operator in the CFT whose vev is given by β has conformal dimension Δ_- in the UV, and therefore the dimension of f is given by

$$[f] = d - 2\Delta_- = 2\nu. \quad (7)$$

The unitarity condition and the requirement of preserving conformal symmetry in the UV, i.e. the perturbation is irrelevant in the UV, demand $|\nu| < d/2$, or $-d^2/4 \leq m^2 < 0$ [20].

These boundary terms are multi-trace perturbations from the point of view of the dual CFT, and have been first studied in e.g. [11–17].

The scalar Klein-Gordon equation can be readily solved in momentum space of the flat directions. For given d -momentum k^μ , $x^{d/2} J_\nu(kz)$ is a solution to the homogeneous version of equation (2) i.e. without the δ -function source, where $J_\nu(kz)$ is a Bessel function of order ν and $k = |k^\mu|$. One can easily solve for the Green's function making use of the completeness of Bessel functions for given boundary condition at $z \rightarrow 0$. Instead of piecing two solutions defined for $z_1 > z_2$ and $z_1 < z_2$ as in [21], we can express $G_{\Delta_-}(z_1, z_2, k)$ as the following integral [22]

$$G_{\Delta_-}(z_1, z_2, k) = \int_0^\infty d\Lambda \frac{\Lambda}{\Lambda^2 + k^2} J_{\nu,f}(\Lambda, z_1) J_{\nu,f}(\Lambda, z_2), \quad (8)$$

where

$$J_{\nu,f}(\Lambda, z) \equiv N(\Lambda) [A(\Lambda) J_\nu(\Lambda z) + B(\Lambda) J_{-\nu}(\Lambda z)] \quad (9)$$

is the linear combination of both Bessel functions of order ν , properly chosen to satisfy the boundary conditions (6) at $z \rightarrow 0$. It is straightforward to check that we must choose

$$A = 1, \quad B = \frac{(2\Lambda)^{2\nu} \Gamma[1 - \nu]}{f \Gamma[1 + \nu]}, \quad (10)$$

and the corresponding normalization factor

$$N(\Lambda)^2 = \frac{1}{1 + 2B(\Lambda) \cos(\nu\pi) + B(\Lambda)^2}, \quad (11)$$

such that $J_{\nu,f}$ is normalized in the sense that

$$\int_0^\infty z \Lambda_1 J_{\nu,f}(\Lambda_1, z) J_{\nu,f}(\Lambda_2, z) dz = \delta(\Lambda_1 - \Lambda_2). \quad (12)$$

The expression for $N(\Lambda)$ can be justified most easily by checking the asymptotic forms of J_ν and $J_{-\nu}$, or using the orthogonality properties of the standard Bessel functions.

One could check, as we demonstrate in Appendix A, that this representation agrees with the bulk-to-bulk propagator in [21] obtained in the Euclidean signature, which simply means k^2 is positive in (8). The virtue of making use of this representation will be made manifest when we begin computing loop corrections, where we are spared of the difficulty of dealing with the step function $\Theta(z - z')$ present in the representation in [21]. Since we are interested in Lorentzian signature we will have to specify precisely the causal structure of our propagators. In the following analysis, we will consider loop corrections to a retarded correlation function. Feynman, or time-ordered, propagators, as discussed in [23], can be obtained by a simple $i\varepsilon$ prescription. Namely, one makes the replacement

$$\frac{1}{\Lambda^2 + k^2} \rightarrow \frac{1}{\Lambda^2 + k^2 - i\varepsilon}. \quad (13)$$

Similarly when we consider retarded (advanced) propagators, we will put all the poles of k_0 in the lower (upper) half complex k_0 plane.

III. 1-LOOP CORRECTION TO GAUGE TWO-POINT CORRELATION AND CUTKOSKY RULES IN ADS SPACE

We are interested in this paper the charged scalar 1-loop correction to the boundary-boundary correlator of a $U(1)$ gauge field in AdS_{d+1} . According to the standard AdS/CFT correspondence, a $U(1)$ conserved current in a d -dimensional CFT is generally dual to a $U(1)$ gauge field, or the bulk “photon” in a $d + 1$ dimensional AdS space. One can view that the CFT lives on the AdS boundary and there is the relation

$$\langle j_\mu j_\nu \rangle_{\text{CFT}_d} = \langle A_\mu A_\nu \rangle_{\partial \text{AdS}_{d+1}}. \quad (14)$$

We will be considering, on the AdS side, loop corrections to this correlator from a charged minimally coupled scalar field that satisfies the general mixed boundary condition explained above. On the CFT side, these loop corrections correspond to $1/N^2$ correction to $\langle j_\mu j_\nu \rangle$.

A. Interaction vertices and the photon propagator

Let us first briefly review the photon boundary-to-bulk propagator at tree level and its interaction with the complex scalar. For convenience, we will work in the gauge

$$A_z = 0 \quad \partial_\mu A^\mu = 0. \quad (15)$$

In Euclidean signature, the boundary-to-bulk propagator satisfying the usual Dirichlet boundary condition at $z = 0$ in this gauge is [23–26]

$$A_\mu(z, p) = J_\mu^\perp(p) \frac{(pz)^{d/2-1} K_{(d/2-1)}(pz)}{(p\epsilon)^{d/2-1} K_{(d/2-1)}(p\epsilon)}, \quad p^\mu J_\mu^\perp(p) = 0. \quad (16)$$

The propagator has been normalised and ϵ is the UV (or boundary $z \rightarrow 0$) cut-off. The boundary source J^\perp satisfies the transverse condition as a result of the gauge choice. The Lorentzian propagator can be readily obtained from the Euclidean one by analytic continuation. Depending on whether one is interested in *in-coming* or *out-going* boundary conditions at the horizon $z \rightarrow \infty$, one could replace p by $\pm ip$ accordingly. Since we are computing a retarded correlation function, we will take one external leg to satisfy in-going boundary conditions, whereas the other leg should be the complex conjugate [27].

There are two types of interaction vertices between minimally coupled charged scalars and photons. The 3-point and the 4-point vertices are given by

$$\begin{aligned} V_{A\phi^\dagger\phi} &= \int d^d x dz \sqrt{-g} g^{\mu\nu} (-i) A_\mu (\phi^\dagger \partial_\nu \phi - \phi \partial_\nu \phi^\dagger) \\ V_{AA\phi^\dagger\phi} &= \int d^d x dz \sqrt{-g} g^{\mu\nu} A_\mu A_\nu \phi^\dagger \phi. \end{aligned} \quad (17)$$

Given that the sources J^\perp in our gauge have to satisfy the transverse condition, only the component satisfying the Ward-identities $p^\mu \langle j_\mu j_\nu \rangle = 0$ would contribute. For general external momenta p the 1-loop contribution is not expected to satisfy the d -dimensional flat-space Ward identities for the boundary theory. When p approaches zero however, the photon wave function becomes independent of the radial coordinate and the integral over the radial position of the vertices yield simple delta functions. In this limit the amplitude is reduced to a d -dimensional calculation, and can be shown to be transverse after all 1-loop diagrams are combined as usual in flat-space.

B. 1-loop diagrams and the Cutkosky rules in AdS space

Now we turn to the 1-loop correction to the boundary-boundary correlator of the bulk photon. Our main objective is to find their imaginary contributions. There are two loop diagrams relevant to the current study as shown in figure (1). Both diagrams contribute to the $1/N^2$ correction to the current-current correlation on the CFT side. Let us

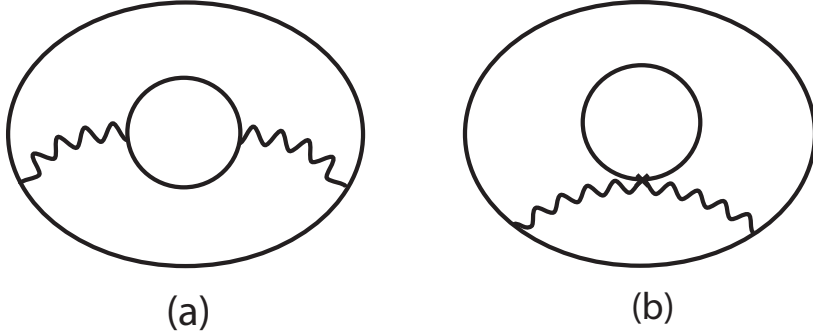


FIG. 1.

denote the contribution from the diagram (a) as $\langle j_\mu j_\nu \rangle'$. Using the propagator (8) and (63), this diagram boils down to the following integral

$$\langle j_\mu(-p) j_\nu(p) \rangle' = \frac{-P_{\mu\alpha}(p) P_{\nu\beta}(p)}{\epsilon^{d-2} |K_{d/2-1}(ip\epsilon)|^2} \int \frac{d^d k d\Lambda_1 d\Lambda_2}{(2\pi)^d} \frac{(p-2k)_\alpha (p-2k)_\beta \Lambda_1 \Lambda_2 |I_{\nu,f}(ip, \Lambda_1, \Lambda_2)|^2}{(\Lambda_1^2 + k^2)_A (\Lambda_2^2 + (p-k)^2)_R}, \quad (18)$$

where we have included the only non-vanishing contribution to this loop, involving the product of a retarded and an advanced scalar propagator. Our readers are reminded of the respective pole structure of the propagators by the subscripts R and A above. We imposed the gauge condition by introducing the usual projection operator

$$P_{\mu\nu}(p) = \eta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}. \quad (19)$$

We have also defined the “vertex function” $I_{\nu,f}$ as

$$I_{\nu,f}(p, \Lambda_1, \Lambda_2) \equiv \int_0^\infty dz z^{\frac{d}{2}} K_{\frac{d}{2}-1}(pz) J_{\nu,f}(\Lambda_1, z) J_{\nu,f}(\Lambda_2, z), \quad (20)$$

which can be read-off directly from equation (17) expressed in momentum space for the d flat directions.

More generally, it is possible that the two scalars propagating in the loop satisfy different mixed boundary conditions, in which case one would have to generalize the “vertex function” in the loop-integral as

$$I_{\nu,f_1,f_2}(p, \Lambda_1, \Lambda_2) \equiv \int_0^\infty dz z^{\frac{d}{2}} K_{\frac{d}{2}-1}(pz) J_{\nu,f_1}(\Lambda_1, z) J_{\nu,f_2}(\Lambda_2, z). \quad (21)$$

This function has an analytic expression that will be discussed further below. At the moment, we just note that, following immediately from the properties of the Bessel functions,

$$I_{\nu,f_1,f_2}(e^{i\frac{\pi}{2}}p, \Lambda_1, \Lambda_2) = I_{\nu,f_1,f_2}^\dagger(e^{-i\frac{\pi}{2}}p, \Lambda_1, \Lambda_2), \quad (22)$$

which we have already made use of in equation (18).

Clearly, the integral would suffer from divergence near the boundary $z \rightarrow 0$, interpreted as a UV divergence, if $2|\nu| > 1$. To avoid complications, we will restrict our attention in this paper to $|\nu| < \frac{1}{2}$.

Written in this form, integral (18) can be conveniently interpreted in the d -dimensional language as the total contribution to the 1-loop correction of $\langle A_\mu A_\nu \rangle$ from all pairs of KK-scalars of masses Λ_1 and Λ_2 . These KK-scalars form a continuous infinite tower whose mass take all values from 0 to infinity, as should be expected since the z -dimension is non-compact. Their contribution to the loop is integrated over, convoluted by a non-trivial “vertex function”, given by $I_{\nu,f}(ip, \Lambda_1, \Lambda_2)$, or more generally I_{ν,f_1,f_2} , describing the mixing of the KK-scalars and their coupling to the gauge field. It should be mentioned that the vertex function I_{ν,f_1,f_2} as defined in (20) is not a proper integral and needs to be regularized. Most naturally, it is regularized by adding to the external momentum p a small positive imaginary part $i\varepsilon$ so that (20) becomes well-defined since its integrand decays exponentially as $z \rightarrow +\infty$, and then taking the $\varepsilon \rightarrow 0$ limit. The result is in general finite and well-defined, but may contain various singularities depending on the values of $(p, \Lambda_1, \Lambda_2)$. Some of the singularities are severe enough that they can render the loop integral (18) ill-defined. Below, we will devote a significant part of this paper discussing these singularities and the regularization of the loop integral.

In this d -dimensional language, one can readily obtain the imaginary part of the loop (18) using the Cutkosky Rules, or the analog of the “optical theorem” for S-matrices. At the level of tree-diagrams this has been considered in [23]. Analyzing the pole structure for the product of retarded and advanced propagators in the same vain shows that the imaginary part can be evaluated by putting the propagators in the loop on-shell, and, as in Feynman loops, only the physical poles would contribute. The cut diagram corresponds to the amplitude of a single “photon” decaying into two KK-scalars of masses Λ_1 and Λ_2 respectively. One then finds the modulus of the amplitude and integrates it over the phase space allowed by the kinematics as well as all positive values of Λ_1 and Λ_2 . Equivalently, one can replace the scalar propagators $[(\Lambda_1 + k^2)(\Lambda_2 + (p - k)^2)]^{-1}$ by their on-shell conditions, i.e. $-(2\pi)^2 \delta(\Lambda_1^2 + k^2) \delta(\Lambda_2^2 + (p - k)^2)$, and carry out the remaining integral. The presence of the two δ -functions confines the integral into a compact region in the phase space. In fact the result is only non-vanishing if the incoming momentum p is time-like and $\sqrt{p_0^2 - |\mathbf{p}|^2} \geq \Lambda_1 + \Lambda_2$.

Consider the simple example where $p_\mu = (p_0, 0, \dots, 0)$ and evaluate the spatial components of the loop diagram. By rotational symmetry and via the Cutkosky rules, we find

$$\begin{aligned} & \int \frac{d^d k}{(2\pi)^{d-2}} k_i k_j \delta(\Lambda_2^2 + (p - k)^2) \delta(\Lambda_1^2 + k^2) \\ &= \delta_{ij} \frac{\Omega_{d-2}}{2(d-1)(2p_0)^d (2\pi)^d} ((p_0^2 - (\Lambda_1 + \Lambda_2)^2)(p_0^2 - (\Lambda_1 - \Lambda_2)^2))^{(d-1)/2} \\ &= \delta_{ij} \frac{\Omega_{d-2} p_0^{d-2}}{8(d-1)(4\pi)^{d-2}} \left((1 - (\tilde{\Lambda}_1 + \tilde{\Lambda}_2)^2)(1 - (\tilde{\Lambda}_1 - \tilde{\Lambda}_2)^2) \right)^{(d-1)/2}, \end{aligned} \quad (23)$$

where Ω_m is the volume of an m -sphere and $\tilde{\Lambda}_{1,2} \equiv \Lambda_{1,2}/p$. The time-time component of the momentum integral similarly evaluates to

$$\begin{aligned} & \int \frac{d^d k}{(2\pi)^{d-2}} (p_0 - 2k_0)^2 \delta(\Lambda_2^2 + (p - k)^2) \delta(\Lambda_1^2 + k^2) \\ &= \frac{\Omega_{d-2} p_0^{d-2}}{8(4\pi)^{d-2}} (\tilde{\Lambda}_1^2 - \tilde{\Lambda}_2^2)^2 \left((1 - (\tilde{\Lambda}_1 + \tilde{\Lambda}_2)^2)(1 - (\tilde{\Lambda}_1 - \tilde{\Lambda}_2)^2) \right)^{(d-3)/2}. \end{aligned} \quad (24)$$

Putting the pieces together, we are left with the integrals of $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ only. By the AdS/CFT dictionary, we find

$$\begin{aligned} \text{Im} \langle j_i(-p_0) j_j(p_0) \rangle' &= \frac{\delta_{ij} \Omega_{d-2} p_0^{d+2}}{8(d-1)(4\pi)^{d-2} \epsilon^{d-2} |K_{d/2-1}(ip_0\epsilon)|^2} \\ &\cdot \int_0^1 d\tilde{\Lambda}_1 \int_0^{1-\tilde{\Lambda}_1} d\tilde{\Lambda}_2 \tilde{\Lambda}_1 \tilde{\Lambda}_2 H(\tilde{\Lambda}_1, \tilde{\Lambda}_2)^{(d-1)/2} |I_{\nu,f}(i, \tilde{\Lambda}_1, \tilde{\Lambda}_2)|^2, \end{aligned} \quad (25)$$

where we defined

$$H(x, y) = [1 - (x + y)^2] [1 - (x - y)^2] \quad (26)$$

and will refer to it as the “phase volume” factor. The correlator $\langle j_0(-p_0) j_0(p_0) \rangle$, on the other hand, is killed off by the projector $P_{\mu\nu}(p)$ for the particular external momentum $(p_0, 0, \dots, 0)$ we have chosen here.

As a consistency check, consider the simple case where the boundary term is set to zero i.e. $f = 0$, where conformal symmetry is expected to be preserved. In that case the p_0 dependence of $I_{\nu,f}$ can be completely taken out, which is simply $I_{\mu,0} \sim p^{-1-d/2}$. Using also the fact that $\lim_{\epsilon \rightarrow 0} K_{\frac{d}{2}-1}(\epsilon p) \sim (\epsilon p)^{-\frac{d}{2}-1}$, the correlator is therefore given by,

$$\langle j_i(-p_0) j_j(p_0) \rangle' = \delta_{ij} \zeta p_0^{d-2}, \quad (27)$$

where ζ is a p independent constant obtained from the rest of the integral. The p_0 dependence of the correlator is exactly what is expected from the conformal dimension of a vector current in a CFT in d -dimensions.

In the case where f is non-zero, the simple scaling behaviour is disturbed and conformal invariance broken. It is manifest that in this case the external momentum p appears as an additional parameter in the loop integral.

We make a comment regarding the loop diagram (b) in figure (1). It is necessary to include this loop to recover gauge invariance in the bulk. But the external momentum in this loop factors out completely and it does not carry any imaginary part, as in the case in flat-space. To extract the imaginary part this diagram can be safely ignored, and its real part can be inferred via gauge invariance from diagram (a). For this reason, we will omit this loop diagram in what follows.

The picture outlined above is mostly standard. One is easily tempted to assume that the loop integral can be evaluated without any obstacles. There, however, remains a quite serious difficulty. As we have already alluded to, it turns out that the “vertex function” I_{ν,f_1,f_2} is not a regular function for all values of Λ_1 and Λ_2 . It contains various divergences of different orders. Expressed in the 2-dimensional space spanned by $\{\Lambda_1, \Lambda_2\}$ the region where I_{ν,f_1,f_2} becomes singular is not isolated but form boundaries, and it diverges sufficiently fast near those boundaries that it can cause the loop integral to diverge, even after the vanishing of the “phase volume” factor $H^{(d-1)/2}$ in this limit is taken into account. This divergences is a new type of divergence, different from the usual UV divergence which still comes about from the remaining $d^d k$ integral. To compute the imaginary part of the 1-loop correction and obtain a physical answer, one must first understand the singularity structures of I_{ν,f_1,f_2} and regularize the loop accordingly. This is the topic we turn to in the next subsection.

C. The divergence of the vertex function and the 1-loop integral

Some discussions on the divergence properties of the “vertex function” I_{ν,f_1,f_2} are in order. We have mentioned that as $f = 0, \infty$, the p dependence of $I_{\nu,0}$ or $I_{\nu,\infty}$ can be completely taken out and apart from an overall power-law dependence of p , the rest, including any possible poles of course, depends on the ratio $\tilde{\Lambda}_{1,2} \equiv \Lambda_{1,2}/p$ only. When $f_{1,2} \neq 0$, there is no longer such a simple scaling behavior of I_{ν,f_1,f_2} . However, as it’s explained later in this subsection as well as in Appendix B, as far as the divergence of I_{ν,f_1,f_2} is concerned, a simple scaling behavior still arises and its singularity structures depend on the ratios $\tilde{\Lambda}_{1,2}$ and $\tilde{f} \equiv f/p$ only. Therefore, without loss of generality, we will set $p = 1$ and assume $\tilde{\Lambda}_{1,2} = \Lambda_{1,2}$ in this subsection, ignoring the scaling of f whenever it’s unimportant.

We will outline the main point and provide the essential results in this section, and leave the full details in Appendix B for those who are interested.

Impressively, Bailey [28] evaluated the following integral involving a power and three Bessel’s functions and found an analytic answer:

$$\begin{aligned} \int dz z^{\lambda-1} K_\rho(cz) J_\mu(az) J_\nu(bz) &= \frac{2^{\lambda-2} a^\mu b^\nu \Gamma[1/2(\lambda + \mu + \nu + \rho)] \Gamma[1/2(\lambda + \mu + \nu - \rho)]}{c^{\lambda+\mu+\nu} \Gamma[\mu+1] \Gamma[\nu+1]} \\ &\cdot F_4\left[\frac{1}{2}(\lambda + \mu + \nu - \rho); \frac{1}{2}(\lambda + \mu + \nu + \rho); \mu+1; \nu+1; -a^2/c^2; -b^2/c^2\right]. \end{aligned}$$

Here, F_4 is the Appell hypergeometric function, one of the generalized hypergeometric function that, apart from four parameters fixed in the current case by the coefficients λ, ρ, μ and ν , depends on two complex variables. For the loop integral that we wish to compute in the Lorentzian signature, we need to analytically continue Bailey's result, substituting c by ic , and obtain the following identity:

$$I \equiv \int_0^{+\infty} z^{\frac{d}{2}} K_{\frac{d}{2}-1}(iz) J_\mu(\Lambda_1 z) J_\nu(\Lambda_2 z) dz$$

$$= \frac{\Gamma\left[1 + \frac{1}{2}(\mu + \nu)\right] \Gamma\left[\frac{1}{2}(d + \mu + \nu)\right] \Lambda_1^\mu \Lambda_2^\nu}{(i)^{d/2+1+\mu+\nu} \Gamma(\mu+1) \Gamma(\nu+1)} F_4\left[1 + \frac{1}{2}(\mu + \nu), \frac{1}{2}(d + \mu + \nu); \mu + 1, \nu + 1; \Lambda_1^2, \Lambda_2^2\right]. \quad (28)$$

This equation, obtained by analytically continuing the convergent integral (28), agrees with the regularization scheme for I_{ν, f_1, f_2} explained earlier where one endows p with a small positive imaginary part that is taken to zero in the end. Within the region $|\Lambda_1| + |\Lambda_2| < 1$, the Appell hypergeometric function has a series expansion which fails to converge as $|\Lambda_1| + |\Lambda_2| \rightarrow 1$. A simple observation of the asymptotics of the integrand in (28) when $z \rightarrow \infty$ leads to the same conditions for I to be well defined. In the current context, both $\Lambda_{1,2}$ are real and positive, which allows us to consider them as the mass of the KK-scalars, and the condition $\Lambda_1 + \Lambda_2 \leq 1$ appears to be nothing other than the consequence of energy-momentum conservation, i.e., a particle can only “decay” into two scalars whose masses add up to a value smaller than the proper energy of the decaying particle. But the fact that I approaches infinity as $\Lambda_1 + \Lambda_2 \rightarrow 1$ is somewhat problematic and a regularization scheme is needed.

To this end, we need to understand exactly how the Appell hypergeometric function F_4 diverges as $\Lambda_{1,2}$ approaches the convergence boundary. It is sufficient to investigate the asymptotics of the integrand in (28) as far as the singularities are concerned. We assume that the values of d, ν and μ are chosen properly such that the integrand is regular at any finite value of z , or if it does contain any singularity at $z < +\infty$, it is a singularity that can be integrated across and leads to no singular behavior of I . Given such assumptions, the only possible source for I to diverge is when the integral is carried out all the way toward $z \rightarrow +\infty$. In the region that z is large, we can approximate the Bessel functions by their asymptotic expansions and formally write

$$I \sim \frac{1}{2^{\frac{d}{2}} \sqrt{\pi} \Lambda_1 \Lambda_2} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} \int_0^{\infty} dz \frac{\left(\frac{d}{2} - 1, n - k - l\right) (\mu, k)(\nu, l)}{(2z)^{n+\frac{3-d}{2}} \Lambda_1^k \Lambda_2^l} \left(e^{-i(\theta_\mu + \theta_\nu + \frac{3\pi}{4})} i^{2(k+l)-n} e^{-i[1-(\Lambda_1 + \Lambda_2)]z} + \dots \right), \quad (29)$$

where $\theta_\mu \equiv \frac{\mu\pi}{2}$ and the ellipsis represents similar terms that involve other combinations of $1 \pm \Lambda_1 \pm \Lambda_2$, whose full form is given in (B7). The notation (ν, n) is defined as

$$(\nu, n) \equiv \frac{\Gamma(\frac{1}{2} + \nu + n)}{n! \Gamma(\frac{1}{2} + \nu - n)}. \quad (30)$$

If we just carry out the integration term by term, regularizing the oscillatory integrand by lifting the integral contour slightly above the real axis as explained earlier, we obtain:

$$I \sim \frac{1}{2^{\frac{3}{2}} \sqrt{\pi} \Lambda_1 \Lambda_2} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} \frac{\left(\frac{d}{2} - 1, n - k - l\right) (\mu, k)(\nu, l) \Gamma(\frac{d-1}{2} - n)}{2^n \Lambda_1^k \Lambda_2^l} \cdot \left(e^{-i(\theta_\mu + \theta_\nu + \frac{(d-2)\pi}{4})} (-)^{k+l} [1 - (\Lambda_1 + \Lambda_2)]^{n+\frac{1-d}{2}} + \dots \right). \quad (31)$$

Again, the ellipsis represents similar terms that involve other combinations of $1 \pm \Lambda_1 \pm \Lambda_2$ and are given in full in (B9) and (B10).

We must elaborate a bit more on the procedure outlined above. The two formulae just given would be flawed if the “ \sim ” were taken to be equal, because the expansions on the r.h.s. of these “equations” are asymptotic expansions only. At any fixed value of z , the infinite sum does not converge. Usually, integrating an asymptotic expansion term by term is only meaningful if the result is also considered as the asymptotic expansion of the true integral when the lower integral limit approaches infinity. However, since for a given integer N , the sum of the first N terms in the expansion approximates the full integrand with an arbitrarily small error when $z \rightarrow \infty$, if one formally integrates the expansion term by term and drops everything that remain finite as $|\Lambda_1| + |\Lambda_2| \rightarrow 1$ along any trajectory, only a finite number of singular terms remain and they describe precisely the same singularities of the original integral (28). Readers may find it suspicious that integrating the asymptotic expansion from $z = 0$ to $+\infty$ could lead to anything meaningful since it is only a good approximation to the actual integrand when z is sufficiently large. More appropriately, one should choose a cutoff scale L that is large but fixed and separate the infinite integral into two

parts: an integral from 0 to L , and an integral from L to $+\infty$. The first piece necessarily contains no poles of $\Lambda_{1,2}$ because L is finite and the integrand is regular. The second piece must consequently include all the singularities of (28), which are independent of the arbitrarily chosen cutoff scale L . [29] Therefore, as far as the singularities are concerned, one can choose an arbitrary cutoff $L > 0$, replace the integrand by its asymptotic form, and carry out the integration from $z = L$ to ∞ term by term. The resulting expansion contains finite number of terms that diverge as $|\Lambda_1| + |\Lambda_2| \rightarrow 1$. Evaluated sufficiently close to the poles, these divergent terms become independent of L , and setting $L = 0$ is only the most convenient choice. We must emphasize that this method would not in general lead to any useful information regarding the finite part of the original integral. Therefore, the “ \sim ” sign in equation (29) means “equal up to an arbitrary regular function”.

It’s worth noting that exceptional cases do exist when such an analysis leads to stronger results. In particular, if in (28) d is odd and μ and ν are half integers, the asymptotic expansions for Bessel functions of half-integer orders are truncated to finite sums in which case the “ \sim ” can be replaced by the equal sign and the equations are exact. Consequently, for $d = 3$, one finds examples where a full analytic result can be obtained, as presented in section IV B.

In the conformal limit, we take $\mu = \nu$ and readily find the singular terms of $I_{\nu,\infty}$ or $I_{\nu,0}$ from equation (31) when $1 - (\Lambda_1 + \Lambda_2) \rightarrow 0$. For general “mixed” boundary conditions we consider, $f \neq 0$ and the dependence of I_{ν,f_1,f_2} on p is more complicated. But the asymptotic behavior of $J_{\nu,f}$ as $z \rightarrow +\infty$ is quite simple and essentially identical to that of standard Bessel functions $J_{\pm\nu}$, except for a simple phase shift explained in details in Appendix B. We can therefore follow the same procedure and easily obtain the most important result of this section: the singularities of $|I_{\nu,f_1,f_2}|^2$ within the domain $0 \leq \Lambda_1, \Lambda_2 \leq 1$ are described by

$$|I_{\nu,f_1,f_2}(i, \Lambda_1, \Lambda_2)|^2 \sim \frac{\Gamma\left(\frac{d-1}{2}\right)^2}{8\pi\Lambda_1\Lambda_2} \left\{ \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{d-1}} + \left[\frac{d-1}{2} - \frac{2(\nu, 1)}{d-3} \left(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2} \right) \right] \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{d-2}} \right\} + O\left([1 - (\Lambda_1 + \Lambda_2)]^{-(d-1)/2}\right), \quad (d > 3). \quad (32)$$

We’ve arrived at a pleasant surprise, which is probably physically well expected: the leading first and the second divergence of $|I_{\nu,f_1,f_2}(i, \Lambda_1, \Lambda_2)|^2$ are independent of the parameter f that describes the mixed boundary condition for the scalar field ϕ . The leading divergence is furthermore independent of its bulk mass parameterized by ν . The remaining singular terms, on the other hand, do in general depend both on f and ν . But fortunately, the orders of those “true” singularities are equal or less than $(d-1)/2$, just sufficiently low to be suppressed by the phase volume factor

$$H^{(d-1)/2} = \{[1 - (\Lambda_1 + \Lambda_2)][1 + \Lambda_1 + \Lambda_2][1 - (\Lambda_1 - \Lambda_2)][1 - (\Lambda_2 - \Lambda_1)]\}^{(d-1)/2}, \quad (33)$$

which contains precisely all four combinations of $(1 \pm \Lambda_1 \pm \Lambda_2)$ to the power of $(d-1)/2$. Hence, the f -dependent imaginary part of the 1-loop integral (25) is finite and can be evaluated free of any pathologies. This conclusion holds true for any $d \geq 2$.

It should be noted once again that, in (31), only terms that become singular as $1 - (\Lambda_1 + \Lambda_2) \rightarrow 0$ should be kept and the rest must be ignored, unless the asymptotic expansion is known to converge. For example, when $d < 4$, only the leading divergent term in (31) is really there. The $1/[1 - (\Lambda_1 + \Lambda_2)]^{(d-3)/2}$ term and therefore the second term in expansion (32) do not exist. Similarly, terms as $1/(1 - \Lambda_1 - \Lambda_2)^{(d-5)/2}$ in (31) is to be discarded if $d < 6$. Precisely as the method fails, the coefficients in expansion (32) become singular.

We remind the readers that for general values of $\Lambda_{1,2}$, I_{ν,f_1,f_2} contains other singularities whenever $1 \pm \Lambda_1 \pm \Lambda_2 \rightarrow 0$ and an analogous expansion can be obtained in the same manner near each of them as shown in Appendix B [30]. In fact there’s a natural way to understand why all those singularities are essentially the same. We take this chance to mention some other interesting properties of the vertex function which might be useful for a full calculation of 1-loop integrals in AdS space. For brevity, we will ignore the differences between photon and scalar and consider a bulk to bulk 1-loop integral [31], in which the following vertex function appears:

$$I(p, \Lambda_1, \Lambda_2) = \int_0^\infty dz z^\alpha J_\nu(pz) J_\nu(\Lambda_1 z) J_\nu(\Lambda_2 z), \quad (34)$$

where α and ν depend on the details of the theory. We’d like to point out that if one can ignore the subtleties related to the singular behavior of the integrand at $z \rightarrow \infty$, formally this function I is defined on \mathbb{CP}^2 if all variables are complex numbers, apart from a trivial power-law factor. This is because one can always absorb an overall scaling of $(p, \Lambda_1, \Lambda_2) \rightarrow (\lambda p, \lambda \Lambda_1, \lambda \Lambda_2)$ by redefining $z \rightarrow \lambda^{-1} z$ together with a change of integral contour which always connects 0 to ∞ in the complex plane. If we restrict ourselves to the real space, one can only absorb such a scaling if $\lambda > 0$, so I is defined on S^2 . Let us focus on the case when p and $\Lambda_{1,2}$ are real and positive. A simple scaling of z immediately allows us to set $p = 1$. We find there are four different regions in the first quadrants in the (Λ_1, Λ_2) plane separated

from each other by singular boundaries. They are $I = \{0 \leq \Lambda_1 + \Lambda_2 \leq 1\}$, $II = \{1 \leq \Lambda_1 - \Lambda_2\}$, $III = \{1 \leq \Lambda_2 - \Lambda_1\}$, and $IV = \{1 \leq \Lambda_1 + \Lambda_2, |\Lambda_1 - \Lambda_2| \leq 1\}$. We have been focusing on region I only in the above discussion. But region II and III can be mapped to I by a simple scaling. To go from region II to region I, we just scale $z \rightarrow \Lambda_1 z$ in (34) and define $\tilde{\Lambda}_2 = \Lambda_2/\Lambda_1$, and $\tilde{\Lambda}_1 = 1/\Lambda_1$. It's easily verified that $0 \leq \tilde{\Lambda}_1 + \tilde{\Lambda}_2 \leq 1$, and

$$I(1, \Lambda_1, \Lambda_2) = \tilde{\Lambda}_1^{\alpha+1} I(1, \tilde{\Lambda}_1, \tilde{\Lambda}_2). \quad (35)$$

Similarly, one can go from region III to region I by a simple scaling of $z \rightarrow \Lambda_2 z$, defining $\tilde{\Lambda}_2 = 1/\Lambda_2$ and $\tilde{\Lambda}_1 = \Lambda_1/\Lambda_2$, and verifying that $I(1, \Lambda_1, \Lambda_2) = \tilde{\Lambda}_2^{\alpha+1} I(1, \tilde{\Lambda}_1, \tilde{\Lambda}_2)$. Notice that $I(1, \Lambda_1, \Lambda_2)$ actually vanishes quite fast as $\Lambda_{1,2}$ is large. Region IV is more complicated, but if one is only interested in the asymptotics of the integrand in (34), which dictates the singularities of I as we argued above, one can also map it to region I by a scaling $z \rightarrow (\Lambda_1 + \Lambda_2)z$, define $\tilde{\Lambda}_1 = (1 + \Lambda_1 - \Lambda_2)/[2(\Lambda_1 + \Lambda_2)]$ and $\tilde{\Lambda}_2 = (1 + \Lambda_2 - \Lambda_1)/[2(\Lambda_1 + \Lambda_2)]$, and find that the singularities of $I(1, \Lambda_1, \Lambda_2)$ are the same as those of $I(1, \tilde{\Lambda}_1, \tilde{\Lambda}_2)$ but with $(\tilde{\Lambda}_1, \tilde{\Lambda}_2)$ located in region I now. Therefore all the singularities of I as $|\Lambda_1| + |\Lambda_2| \rightarrow 1$ are related to each other in a simple way.

Just as a demonstration, we show in figure (2) the numerical result for the relative variation of the imaginary part of the 1-loop diagram against the boundary coupling f when $d = 4$ and $\nu = \frac{1}{2}$. The numerical result presented in the plot is subtracted against that when $f = \infty$.

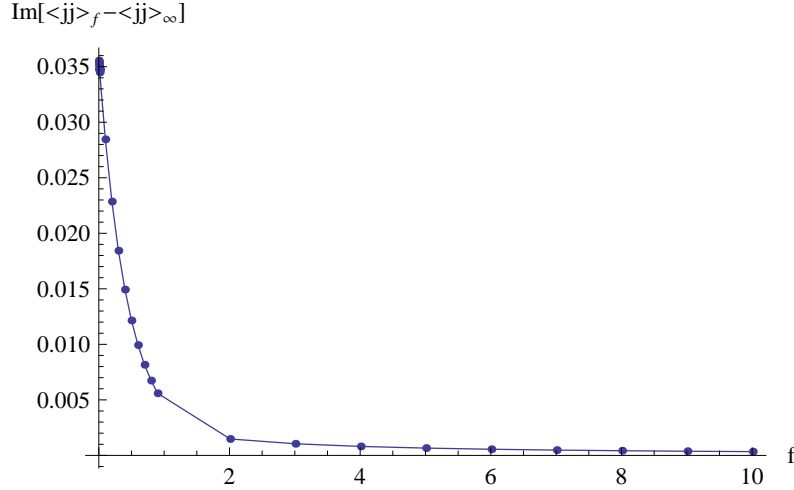


FIG. 2. The relative variation of the imaginary part of the 1-loop integral against coupling f at $d = 4, \nu = \frac{1}{2}$. External momentum being an overall scale is set to one.

D. A few more general observations on the divergences in AdS loops

Thus far we have studied closely loops involving charged scalars coupled to photons and the divergences associated to the vertex. We would like to make a few observations about loops in AdS more generally. We found that this kind of divergence occurring as the vertices approach the AdS horizon is fairly general. Consider for simplicity a ϕ^n vertex of the form

$$\mathcal{V}(\phi) = \int \prod_i^n d^d p_i \int_0^\infty dz \sqrt{-g} \delta^d \left(\sum_i^n p_i^\mu \right) \prod_i^n \phi(p_i, z) \quad (36)$$

expressed in momentum space for the flat directions. Substituting in the integral the radial wave-function of the scalar field of mass m , which, for Neumann boundary condition as an example, is given by $z^{d/2} J_\nu(pz)$, where $p = |p^\mu|$, and taking the near horizon limit, we find

$$\mathcal{V}(\phi) \sim \int \prod_i^n d^d p \delta^d \left(\sum_i^n p_i^\mu \right) \int_0^\infty dz (z^{-d-1}) z^{n \frac{d-2}{2}} \sum_{\varepsilon_i = \pm 1} \cos \left[\sum_i^n (\varepsilon_i p_i z + \varepsilon_i \theta_\nu) \right]. \quad (37)$$

We have used the asymptotic behavior $\lim_{z \rightarrow \infty} J_\nu(pz) \sim \frac{1}{\sqrt{pz}} \cos(pz + \theta_\nu)$. By naive power counting, ignoring the oscillatory factors, this integral is only convergent if

$$\frac{n(d-2)}{2} < d+1, \quad (38)$$

precisely identical to the renormalizability condition one would find via naive counting of engineering dimensions. Notice this analysis is independent of the boundary condition for the scalar since it would only affect the oscillatory factors and leave the power-law dependence intact.

Such a coincidence no longer exists as soon as we consider coupling to fields with higher spins. In the case of photons, the vertex would appear divergent by the above naive power-counting for arbitrary d independently of the renormalizable conditions, because the coupling invariably involves the inverse metric which provides extra factors of z^2 even though the photon propagator is proportional only to $z^{\frac{d-2}{2}} J_{d/2-1}$, and subsequently contribute to the divergence in the horizon limit. This consideration is generally applicable to the coupling of scalars and graviton as well.

It is very important to note that even as the z -integral appears divergent according to the above consideration, it does not, however, necessarily imply that the loop integral is divergent. The vertex integral can be regularized by analytic continuation as we have done in our earlier discussions making use of its oscillatory factors. Using the method introduced in the previous section, one can easily verify that this vertex after being regularized must include poles at $\sum_i^n \varepsilon_i p_i = 0$, where $\varepsilon_i = \pm 1$, and could potentially lead to similar divergences in loop integrals in the collinear limit as $\sum_i^n p_i \rightarrow 0$. Depending on the order of these poles, which must be analyzed case by case following the procedure explained in section III C, this may or may not lead to pathologies. Let us mention, as an example here, the same analysis implies that this extra divergence occurring as the vertex in diagram (b) in figure (1) approaches the horizon is logarithmic if $d = 4$.

We would also like to point out here that using the form of fermion wave-function in AdS as given in [32], we can repeat the above power counting for a fermion-photon vertex as $z \rightarrow \infty$ and find

$$\int^\infty dz \sqrt{-g} A_\mu e_a^\mu \bar{\psi} \Gamma^a \psi \sim \int^\infty dz z^{\frac{d-3}{2}} \times \text{oscillatory factors}, \quad (39)$$

which is exactly the same as in the case of scalars. Here e_a^μ is a simple diagonal choice for the vierbeins, and Γ^a are flat space gamma matrices. It seems possible that in supersymmetric theories these type of divergences are cancelled automatically among fermions and bosons.

Finally, we make a passing comment about potential UV divergence near the boundary $z \rightarrow 0$. We have restricted our attention to scalar fields of mass m such that $\nu = \sqrt{\frac{d^2}{4} + m^2} \leq \frac{1}{2}$ because this is a window of consistent masses such that UV divergence near the boundary is simply non-existent, and curiously this condition is dimensional independent.

It would be interesting to have a better understanding of the physics behind these observations.

IV. RANDOM DISORDER AND THE REPLICA TRICK IN ADS/CFT

We now turn to an interesting application of the results given in the previous section and calculate the effect of random disorder on the transport coefficients using AdS/CFT correspondence.

A. The replica trick and the conductivities

We begin with a brief review of the replica trick. Consider a quantum system described by the action S . The effects of random disorder can be captured by introducing into the action an extra scattering potential δS

$$\delta S = \int d^d x V(x, t) \mathcal{O}(x, t), \quad (40)$$

where $V(x, t)$ is a general spacetime dependent random scattering potential, and \mathcal{O} is some physical operator, such as the charge density etc. In the presence of the random scattering potential term, correlation functions averaged over the random variable V is given by

$$\overline{\langle \mathcal{O}(x_1) \mathcal{O}(x_2) \dots \rangle} = \int D[V] P[V] \left(\frac{\int D\phi e^{-S - \delta S} \mathcal{O}(x_1) \mathcal{O}(x_2) \dots}{\int D\phi e^{-S_0 - \delta S}} \right), \quad (41)$$

where $P[V]$ is the probability distribution of the random potential and the over-line denotes averaging over V . A well-known trick in the condensed matter literature to compute this averaged correlator is called the *replica trick*. The idea is to introduce n copies of the theory concerned such that all operators in the theory are replicated n times and we label them by an extra index $i \in \{1, \dots, n\}$. The partition function of the full theory including the n copies is related to that of the original one by $\mathcal{Z}_n = (\mathcal{Z}_1)^n$. Then the correlation function for a fixed V in the original theory is formally related to the n -replica partition function, treating n as a continuous variable by analytic continuation as

$$\langle \mathcal{O} \mathcal{O} \dots \rangle_V = - \lim_{n \rightarrow 0} \frac{\delta}{\partial J_1} \frac{\delta}{\partial J_1} \dots \frac{1}{n} (e^{n \ln \mathcal{Z}_1} - 1) \Big|_{J_i=0} = - \lim_{n \rightarrow 0} \frac{\delta}{\partial J_1} \frac{\delta}{\partial J_1} \dots \frac{1}{n} \mathcal{Z}_n \Big|_{J_i=0}, \quad (42)$$

where J_i is the current coupled to the replicated operator \mathcal{O}_i . The averaged correlation functions can be readily evaluated as

$$\overline{\langle \mathcal{O} \mathcal{O} \dots \rangle} = - \lim_{n \rightarrow 0} \frac{\delta}{\partial J_1} \frac{\delta}{\partial J_1} \dots \frac{1}{n} \int D[V] e^{\ln P[V]} \mathcal{Z}_n \Big|_{J_i=0} = - \lim_{n \rightarrow 0} \frac{\delta}{\partial J_1} \frac{\delta}{\partial J_1} \dots \frac{1}{n} \tilde{\mathcal{Z}}_n \Big|_{J_i=0}. \quad (43)$$

We have exchanged the order of taking the $n \rightarrow 0$ limit and the integral of V in the first step and defined

$$\tilde{\mathcal{Z}}_n \equiv \int D[V] \prod_i D[\phi_i] e^{-\sum_{i=1}^n (S_i + \delta S_i) + \ln P[V]} \quad (44)$$

in the second. For more complete explanation, see [33] and references therein.

This relation (43) is useful because in the simple but common situation where the random distribution $P[V]$ is simply Gaussian and local in space and time i.e.

$$P[V] = e^{-\frac{V^2(x,t)}{2f}}, \quad (45)$$

the scattering potential V can be readily integrated out, and we have

$$\tilde{\mathcal{Z}}_n = \int \prod_i D[\phi_i] e^{-\sum_{i=1}^n S_i - \frac{f}{2} \int d^d x \sum_{ij} \mathcal{O}^i \mathcal{O}^j}. \quad (46)$$

To study the effects of random disorder in a strongly coupled theory, it is natural to incorporate the replica trick in the context of AdS/CFT correspondence, which is considered in [10, 34]. The spirit of the two studies are very different and in this section we follow [10] and attempt to study the effects of disorder on the conductivity of a general charged system.

In [10] the replicas are introduced literally as n copies of the AdS backgrounds. The random scattering potential in the CFT in the replicated theory is presented by the boundary term

$$\delta S = \int_{\partial \mathcal{M}_{d+1}} d^d x V(x, t) \sum_i \mathcal{O}^i, \quad (47)$$

which gives

$$\delta S = -\frac{f}{2} \int_{\partial \mathcal{M}_{d+1}} d^d x \sum_{i,j} \mathcal{O}^i \mathcal{O}^j, \quad (48)$$

after the potential is integrated out. These boundary terms are precisely those multi-trace perturbations in the dual CFT theory discussed in earlier sections, and dictate the boundary conditions of the corresponding scalar fields in the multiple AdS bulks. They also relate the boundary values of the scalars in different AdS copies and thus connect them together. The modified boundary-boundary and bulk-to-bulk propagators of these scalar fields have been studied in [21] and, in the context of the replica trick, in [10]. It is known that the effect of these boundary perturbations are mediated to other sectors beginning only at 1-loop order [12, 35, 36]. Using the *folding trick* that treats these replicated bosons as if they live in the same AdS space [35] we can thus apply techniques discussed in the previous sections to compute the conductivities in the presence of random disorder.

Now consider again scalar field ϕ with mass m . Recall that its boundary expansion is given by

$$\phi \sim \alpha z^{\Delta_+} + \beta z^{\Delta_-}, \quad \Delta_{\pm} = \frac{d}{2} \pm \sqrt{m^2 + \frac{d^2}{4}}.$$

In the presence of the boundary deformations (48), the scalar fields concerned satisfy nontrivial boundary conditions.

$$\alpha_i = f \sum_j^n \beta_j, \quad (49)$$

where n is the number of replicas introduced. These conditions can be diagonalized by:

$$\tilde{\phi}_l = \sum_j^n a_j^l \phi_j, \quad 0 < l < n, \quad a_j^n = \frac{1}{\sqrt{n}}, \quad \sum_j a_j^{l \neq n} = 0. \quad (50)$$

There is only one linear combination of the fields ϕ_i (corresponding to $\tilde{\phi}_n$ defined above) satisfies the mixed boundary conditions depending on the strength of the disorder f , i.e.,

$$\tilde{\alpha}_n = n f \tilde{\beta}_n, \quad (51)$$

whereas the rest of the fields $\tilde{\phi}_{l \neq n}$ simply satisfy usual Neumann boundary conditions

$$\tilde{\alpha}_{m \neq n} = 0. \quad (52)$$

It would be useful to pick an orthonormal basis for these fields, so that the bulk-to-bulk propagators take a simple form. One convenient choice is

$$a_i^{(n-l)} = \frac{1}{\sqrt{(n-l)(n-l+1)}} \begin{cases} 1, & 1 \leq i \leq (n-l) \\ -(n-l), & i = n-l+1 \\ 0, & i > n-l+1. \end{cases} \quad (53)$$

The bulk-to-bulk propagators of the fields ϕ_i in the original basis would be related to these rotated basis $\tilde{\phi}$ by simple linear combinations. In particular, we are interested in the following propagators

$$\begin{aligned} G^{nn} &= \frac{1}{n} (\tilde{G}^{nn} + (n-1) \tilde{G}^{(n-1)(n-1)}), & G^{ij} &= \langle \phi_i \phi_j \rangle, & \tilde{G}^{ij} &= \langle \tilde{\phi}_i \tilde{\phi}_j \rangle, \\ G^{n(n-1)} &= \frac{1}{n} (\tilde{G}^{nn} - \tilde{G}^{(n-1)(n-1)}). \end{aligned} \quad (54)$$

The bulk-to-bulk propagators of the orthonormal scalar fields $\tilde{\phi}_l$ satisfy the usual Klein-Gordon equations in AdS_{d+1} space, as discussed in the previous sections. Explicitly, we have, in Euclidean signature and in the $n \rightarrow 0$ limit,

$$\begin{aligned} G^{ii}(x, y, k) &= \int d\Lambda \frac{\Lambda}{\Lambda^2 + k^2} \left(J_{-\nu}(\Lambda x) J_{-\nu}(\Lambda y) + \frac{f \Gamma(1+\nu)}{\Gamma(1-\nu)(2\Lambda)^{2\nu}} (J_{\nu}(\Lambda x) J_{-\nu}(\Lambda y) \right. \\ &\quad \left. + J_{-\nu}(\Lambda x) J_{\nu}(\Lambda y) - 2 \cos(\nu\pi) J_{-\nu}(\Lambda x) J_{-\nu}(\Lambda y) \right), \end{aligned} \quad (55)$$

and

$$G^{i,j \neq i}(x, y, k) = \int \frac{\Lambda d\Lambda}{\Lambda^2 + k^2} \left[\frac{f \Gamma(1+\nu)}{\Gamma(1-\nu)(2\Lambda)^{2\nu}} (J_{\nu}(\Lambda x) J_{-\nu}(\Lambda y) + J_{-\nu}(\Lambda x) J_{\nu}(\Lambda y) - 2 \cos(\nu\pi) J_{-\nu}(\Lambda x) J_{-\nu}(\Lambda y)) \right]. \quad (56)$$

The conductance in Lorentzian signature, related to $\langle \bar{j}_i(-p) j_j(p) \rangle$ and $\langle \bar{j}_i(-p) \rangle \langle j_j(p) \rangle$, are thus given by precisely the same 1-loop calculation as detailed in the previous section, except the scalar bulk-to-bulk propagators are replaced by G^{ii} and $G^{i,j \neq i}$ respectively.

Given the notable complication in the form of the propagators (54), one would like to know if the divergence occurring in the collinear limit of the momenta, discussed extensively in the previous section, could actually acquire some n or boundary coupling f dependence, rendering our procedure in extracting the transport properties pathological. Fortunately, one can show that just as before, the divergent terms that are not extinguished by the phase space volume factor are again n and f independent.

To see that, let us rewrite for example the basis rotation:

$$\phi_n = \frac{1}{\sqrt{n}} \tilde{\phi}_n - \sqrt{\frac{n-1}{n}} \tilde{\phi}_{n-1} = \cos \xi \tilde{\phi}_n - \sin \xi \tilde{\phi}_{n-1}, \quad (57)$$

where we remind our readers that the tilde fields are those that diagonalize the boundary conditions. We deliberately rewrite the coefficients in terms of sine and cosine to make it explicit that the squares of these coefficients add up to one. The corresponding propagator is

$$G^{nn} = \cos^2 \xi \tilde{G}^{nn} + \sin^2 \xi \tilde{G}^{(n-1)(n-1)}, \quad (58)$$

and to be explicit,

$$\tilde{G}^{nn} = \int d\Lambda \frac{\Lambda}{\Lambda^2 + k^2} J_{\nu,nf} J_{\nu,nf}, \quad \tilde{G}^{(n-1)(n-1)} = \int d\Lambda \frac{\Lambda}{\Lambda^2 + k^2} J_{\nu,0} J_{\nu,0}, \quad (59)$$

where $J_{\nu,f}$ are defined in (9). Plugging the propagator G^{nn} into the loop including the external photon propagator then leads to the products

$$\begin{aligned} & x^{d/2} y^{d/2} K_{\frac{d}{2}-1}(-ipy) K_{\frac{d}{2}-1}(ipx) \prod_{i=1}^2 [\cos^2 \xi J_{\nu,nf}(\Lambda_i x) J_{\nu,nf}(\Lambda_i y) + \sin^2 \xi J_{\nu,0}(\Lambda_i x) J_{\nu,0}(\Lambda_i y)] \\ & \sim x^{(d-3)/2} y^{(d-3)/2} \exp(ipx) \exp(-ipy) \prod_{i=1}^2 \left[\cos^2 \xi (\cos(\Lambda_i x - \theta_{\nu,nf} - \frac{\pi}{4})) (\cos(\Lambda_i y - \theta_{\nu,nf} - \frac{\pi}{4})) \right. \\ & \quad \left. + \sin^2 \xi (\cos(\Lambda_i x - \theta_{\nu,0} - \frac{\pi}{4})) (\cos(\Lambda_i y - \theta_{\nu,0} - \frac{\pi}{4})) \right], \end{aligned} \quad (60)$$

where we have omitted some overall numerical factors and powers of Λ_i 's which are common to all scalar loops independently of boundary perturbations.

Concentrating on the collinear limit where one actually encounters a divergence i.e. $\Lambda_1 + \Lambda_2$ is close to p , the dominant contribution from the above expressions, after doing the radial integral over x and y is given similar to the discussion leading to (32), by

$$\begin{aligned} & \sim (p - \Lambda_1 - \Lambda_2)^{1-d} \left(\cos^4 \xi e^{-i(2\theta_{\nu,nf} - \frac{\pi}{2})} e^{i(2\theta_{\nu,nf} - \frac{\pi}{2})} + \sin^4 \xi e^{-i(2\theta_{\nu,0} - \frac{\pi}{2})} e^{i(2\theta_{\nu,0} - \frac{\pi}{2})} \right. \\ & \quad \left. + 2 \cos^2 \xi \sin^2 \xi e^{-i(\theta_{\nu,nf} + \theta_{\nu,0} - \frac{\pi}{2})} e^{i(\theta_{\nu,nf} + \theta_{\nu,0} - \frac{\pi}{2})} \right) \\ & = (p - \Lambda_1 - \Lambda_2)^{1-d} (\cos^2 \xi + \sin^2 \xi)^2 = (p - \Lambda_1 - \Lambda_2)^{1-d}. \end{aligned} \quad (61)$$

The dependence on n and f drops out in the leading divergence. One could repeat the exercise for the next leading order divergence to see that again it is n and f independent. Since the expansion in powers of n is completely regular in the propagators, the divergences remain n, f independent even as we expand first in n before doing the integral. We will in fact work out in detail in the following section an analytic example where the result after doing the radial integrals is particularly simple, and explicitly in this example the divergence can be seen to be n, f independent as we have demonstrated via the asymptotic expansion method here.

We extract the dissipative part of the conductivity from the correlation functions via the Kubo formula

$$\sigma(p_0) = \frac{1}{p_0} \text{Im}[\overline{\langle j_i(-p_0) j_j(p_0) \rangle_{\text{retarded}}}] \quad (62)$$

We include in figure (3) a numerical plot of the conductivity against coupling f at $d = 4, \nu = \frac{1}{2}$. The resultant conductivity is regulated, for convenience, by subtracting off a corresponding scalar loop satisfying Dirichlet boundary conditions in a simple un-replicated AdS. One can see that the conductivity decreases as disorder is turned on. This point shall be discussed again in our analytic example in the next section.

We point out that if the random disorder has a non-trivial power function in momentum space, which corresponds to spatial correlation of these impurities, one can replace f by an appropriate function $f(p)$ and the results are straightforwardly generalized by the same procedure.

B. An analytic example in $d = 2 + 1$

At $d = 2 + 1$ and $\nu = \frac{1}{2}$, the result is particularly simple. The Euclidean photon propagator given by (16) for $d = 3$ reduces to

$$A_\mu(z, p) = J_\mu^\perp(p) \frac{(pz)^{d/2-1} K_{\frac{1}{2}}(pz)}{(p\epsilon)^{\frac{1}{2}} K_{\frac{1}{2}}(p\epsilon)}, \quad p^\mu J_\mu^\perp(p) = 0, \quad K_{\frac{1}{2}}(pz) = e^{-pz} \sqrt{\frac{\pi}{2pz}}. \quad (63)$$

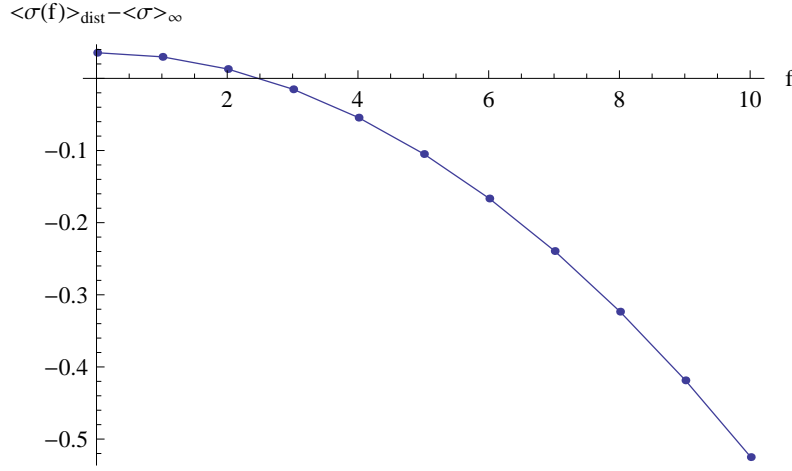


FIG. 3. Dissipative part of the 1-loop contribution to the conductivity against coupling f at $d = 4, \nu = \frac{1}{2}$. External momentum being an overall scale is set to one.

Similarly, scalar propagators proportional to products of $J_{\pm\frac{1}{2}}(\Lambda x)$ become simply products of $\sqrt{2/\pi\Lambda x} \sin \Lambda x$ and $\sqrt{2/\pi\Lambda x} \cos \Lambda x$. More explicitly, (55) becomes in this case

$$G^{ii}(x, y, k) = \int d\Lambda \frac{\Lambda}{\Lambda^2 + k^2} F^{ii}(x, y, \Lambda),$$

$$F^{ii}(x, y, \Lambda) = \left(\frac{2 \cos(x\Lambda) \cos(y\Lambda)}{\pi \Lambda \sqrt{xy}} + \frac{f}{2\pi \Lambda^2 \sqrt{xy}} (\cos(x\Lambda) \sin(y\Lambda) + \sin(x\Lambda) \cos(y\Lambda)) \right). \quad (64)$$

Plugging into the loop integral and denoting the radial coordinates of the vertices x, y , in Lorentzian signature, we have

$$I_{\text{disorder}} = \int dx dy K_{\frac{1}{2}}(ipx) K_{\frac{1}{2}}(-ipy) F^{ii}(x, y, \Lambda_1) F^{ii}(x, y, \Lambda_2)$$

$$= \frac{8(-1 + \tilde{\Lambda}_1^2 + \tilde{\Lambda}_2^2)^2 - \tilde{f}^2(\tilde{\Lambda}_1^4 + (-1 + \tilde{\Lambda}_2^2)^2 - 2\tilde{\Lambda}_1^2(1 + \tilde{\Lambda}_2^2))}{4p^5 \pi \tilde{\Lambda}_1 \tilde{\Lambda}_2 (\tilde{\Lambda}_1^4 + (-1 + \tilde{\Lambda}_2^2)^2 - 2\tilde{\Lambda}_1^2(1 + \tilde{\Lambda}_2^2))}, \quad (65)$$

where again we used the dimensionless quantities

$$\tilde{\Lambda}_i = \frac{\Lambda_i}{p}, \quad \tilde{f} = \frac{f}{p}. \quad (66)$$

It is important to note here that there is again a divergence not cured by the phase space factor in the collinear limit proportional to $(p - \Lambda_1 - \Lambda_2)^{1-d} = (p - \Lambda_1 - \Lambda_2)^{-2}$. However, it is completely independent of the boundary perturbation coupling f . Given the explicit analytic result of the integral, we shall subtract off the divergence directly here.

Now we can compute the dissipative part of the conductivity in the presence of the disorder and find

$$\sigma(p_0)_{1\text{-loop}} = \frac{1}{p_0} \text{Im}[\langle j_i(-p_0) j_j(p_0) \rangle_{1\text{-loop}}]$$

$$= \lim_{\epsilon \rightarrow 0} \epsilon \delta_{ij} \frac{(2\pi)^3 p_0^4}{4(4\pi)^3 |K_{\frac{1}{2}}(ip_0\epsilon)|^2} \int_0^1 d\tilde{\Lambda}_1 \int_0^{1-\tilde{\Lambda}_1} d\tilde{\Lambda}_2 \tilde{\Lambda}_1 \tilde{\Lambda}_2 H(\tilde{\Lambda}_1, \tilde{\Lambda}_2) I_{\text{disorder}}$$

$$= \lim_{\epsilon \rightarrow 0} \epsilon \delta_{ij} \frac{-2(\frac{f}{p_0})^2 + \frac{925}{72} - \frac{\log(2)}{3} + \log(8)}{32p_0(4\pi)^3 |K_{\frac{1}{2}}(ip_0\epsilon)|^2}. \quad (67)$$

The f -independent part is simply the 1-loop correction to the conformal result. The precise value is unimportant, since it depends on how we subtracted the divergence. The f -dependent contribution however is interesting. First of all, it is negative definite, which means it always reduces the conductivity. Secondly, while the loop correction

is only of order N^0 , it is clear that in the low frequency limit $(p_0/f^2)^2 \ll N^{-2}$ and this loop correction would eventually become more important than the tree-level contribution, which is of order N^2 . This is not surprising since the boundary perturbation is deliberately chosen to be relevant in the infrared. This means that to extract result in the deep infrared limit one probably needs to re-sum the loop contributions. This could be the explanation as to how the delta-function at $p_0 = 0$ is corrected even though naively the boundary perturbation only begins to contribute at 1-loop level and appears extremely weak. We would like to pursue this in more detail in future work.

Time independent disorder

When the disorder is time independent, only the component of the scalar $\phi(k, z)$ with $k = (0, k_i)$ acquires a nontrivial boundary condition (6). i.e. the disorder coupling f is frequency dependent and is more precisely given by $f\delta_{p_0,0}$. The disorder therefore only contributes non-trivially if at least one of the two internal bulk-to-bulk propagators is evaluated at exactly zero energy. By the Cutskovsky rules, the imaginary part of the loop is only non-zero when $\Lambda_{1,2}$ can satisfy the “on-shell” conditions that restrict them to be real and non-negative. All these conditions can be satisfied only at one point of the phase space where $k_i = \Lambda_1 = 0$. In other words, by putting the KK-scalar fields on-shell and requiring that one of them has zero energy, its corresponding spatial momentum and effective mass has to be zero. Note however that the boundary disorder coupling f always come in the dimensionless combination $(\Lambda)^{2\nu}/f$. Therefore as soon as we set $\Lambda = 0$ such that the momentum dependent coupling f contributes, its contribution is immediately killed by the $(\Lambda)^{2\nu}$ factor, for any positive ν . The time-independent disorder therefore doesn’t seem to have any effect on the dissipative part of the conductivity, even though the operator is relevant. The only possibility of a non-trivial contribution seems to be the marginal limit where $\nu \rightarrow 0$ and $\Delta_{\mathcal{O}} \rightarrow \frac{d}{2}$. The computation should follow the same logic as presented in the previous sections, although the radial wave-function of the scalar field would involve $J_0(\Lambda x)$ and the boundary expansion (3) has to be modified to include log-terms, in which case the correct linear combinations of Bessel functions satisfying the relevant boundary conditions have to be computed separately. We will leave these interesting possibilities for future work.

V. CONCLUSION AND DISCUSSION

We have studied in the AdS space the 1-loop correction to the boundary-boundary 2-point function of a $U(1)$ gauge field coupled to a complex scalar. The scalar is subjected to quadratic boundary perturbations and satisfies *mixed* boundary condition. These boundary perturbations are related to double-trace perturbations in the dual CFT. While we concentrated our effort toward studying the imaginary part of the diagrams, which is of particular interest physically and can be readily extracted via a simple generalization of Cutkosky rules [23], we found that in Poincare coordinates these loops suffer from extra divergences shared by both the real and imaginary parts of the diagram, arising when the vertices of the diagrams are pushed toward the AdS *horizon* simultaneously and the geodesic distance between them shrinks to zero. When the loop integral is reduced to an integral of both vertex positions along the radial direction which gives rise to a product of two double variable hypergeometric functions (the Apell function F_4), the divergence manifests itself as singularities along the boundary in the phase space where the momenta of the photon and the loop scalar fields become collinear.

Fortunately, we manage to derive an explicit representation for these divergences and discover that they are independent of the boundary couplings, or in other words, the mixed boundary condition for the complex scalar. While F_4 is known only as a power series, and the singularities that concern us occur precisely along the boundary of its convergent domain, we were able to extract the exact properties of its singularities nevertheless using simply the asymptotic expansion of the radial wave-functions of the propagator. It is found that the order of divergences are finite and is related to the spacetime dimension in a simple algebraic way. The coefficients are easily computed as well. Most curiously, the leading divergence turns out to be universal, depending only on the spacetime dimension, whereas the first sub-leading divergence depends upon the spacetime dimension and the bulk mass of the scalar field only. The method presented can be applied to analyze the singularity structures of other functions that have a similar integral form, such as similar vertices in other curved spacetime background.

These results are readily applicable to certain physical problems. In particular, we applied AdS/CFT correspondence to condensed matter systems and studied the effects of random disorder on conductivity, using the replica trick [10]. The effect of the disorder in this setup is captured by double-trace boundary perturbations, corresponding precisely to mixed boundary condition on the AdS side for the complex scalar. The dissipative part of the current-current correlation in the CFT is given by the imaginary part of the photon boundary-boundary correlator in the AdS space, which begins to exhibit the effect of the disorder from the photon coupling to the particular charged scalar at 1-loop. Since the singularities of the relevant loop integral are found to be universal, we are able to subtract the results of the conductivities obtained at various disorder coupling by the corresponding conformal result, and obtain a finite

difference. We found that in general the disorder reduces the conductivity. Also, since the disorder we consider is relevant in the infrared, we find that generally in the extreme low frequency limit, the loop correction begins to overwhelm the planar contribution, suggesting the need for re-summation, but allows for the possibility of the removal of the delta-function in the conductivity known to exist at zero frequency due to momentum conservation[4] even though the effect of the disorder appears at first sight to be loop-suppressed.

Let us mention a couple of words on some other interesting observations and conjectures. We found that the same divergence in the limit as loop vertices approach the horizon is quite general in loop integrals in AdS space, particularly when fields of higher spins, such as photons and gravitons are involved, although they most likely require more intriguing regularizations. However, for loops involving only scalar ϕ^n vertices, the absence of divergence is correlated to renormalizability of the theory concerned. It is also interesting that the photon-fermion-fermion vertex displays a singularity near the horizon with precisely the same power law behavior as the photon-scalar-scalar vertex. It would be interesting to understand the precise physics of these divergences. In the Poincare patch where a mass gap is absent, we have, correspondingly in the bulk a continuum spectrum of “Kaluza-Klein” mass Λ that can be arbitrarily small, which might be intimately related to the collinear divergence in the decay amplitude of the photon as we have observed.

Yet, it is suggested in [37] that in real-time computations one should build wave-functions such that they vanish at the horizon. This would mean that waves could only propagate as wave-packets rather than as momentum eigenstates along isometry directions. It is probable that the divergence can be cured by building a wave packet that vanishes in the horizon limit but asymptotes to a plane wave near the boundary. However, since the regular part of the diagram in the collinear limit is heavily suppressed by the phase-space volume, we expect that our results should not be too sensitive to the near horizon behavior of the wave-functions. We leave these important and rigorous endeavor for future work.

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Appendix A: Contour integral of the scalar bulk-to-bulk propagator

In the Euclidean signature, the scalar bulk-to-bulk propagator is given by:

$$\int_0^{+\infty} \frac{\Lambda d\Lambda}{\Lambda^2 + k^2} \frac{[J_\nu(\Lambda x) + \tilde{f}\Lambda^{2\nu}J_{-\nu}(\Lambda x)][J_\nu(\Lambda y) + \tilde{f}\Lambda^{2\nu}J_{-\nu}(\Lambda y)]}{1 + \tilde{f}^2\Lambda^{4\nu} + 2\tilde{f}\Lambda^{2\nu}\cos\nu\pi}. \quad (\text{A1})$$

We denoted $\tilde{f} = (2\Lambda)^{2\nu}\Gamma(1-\nu)/[\Gamma(1+\nu)]$ for brevity.

We can evaluate this propagator by replacing it by a contour integral. Let's assume $x > y$ for the moment. Notice that

$$J_\nu(\Lambda x) + \tilde{f}\Lambda^{2\nu}J_{-\nu}(\Lambda x) = \frac{1}{\pi i}(e^{-\nu\pi i/2} + \tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2})K_\nu(\Lambda x e^{-\pi i/2}) - \frac{1}{\pi i}(e^{\nu\pi i/2} + \tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2})K_\nu(\Lambda x e^{\pi i/2}) \quad (\text{A2})$$

and

$$1 + \tilde{f}^2\Lambda^{4\nu} + 2\tilde{f}\Lambda^{2\nu}\cos\nu\pi = (\tilde{f}\Lambda^{2\nu} + e^{\nu\pi i})(\tilde{f}\Lambda^{2\nu} + e^{-\nu\pi i}), \quad (\text{A3})$$

so the original integral is also given by

$$\int_0^{+\infty} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \left[\frac{K_\nu(-i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} - \frac{K_\nu(i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \right] \cdot (\text{y terms}). \quad (\text{A4})$$

We define

$$I_+ = \int_0^{+\infty} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(-i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} \cdot (\text{y terms}), \quad (\text{A5})$$

and

$$I_- = - \int_0^{+\infty} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \cdot (\text{y terms}). \quad (\text{A6})$$

Let us also denote C_+ the closed contour in the complex plane that consists of the entire real axis and the infinitely large semicircle in the *upper* half plane, and C_- the opposite contour that consists of the real axis and the infinitely large semicircle in the *lower* half plane.

It's readily verified that if one substitutes

$$\Lambda \rightarrow \Lambda e^{\pi i} \quad (\text{A7})$$

one finds

$$I_+ \rightarrow -I_- . \quad (\text{A8})$$

Similarly, upon substituting Λ by $\Lambda e^{-\pi i}$, one finds I_- becomes $-I_+$. The two phase choices just mentioned, however, are mutually exclusive and therefore the full integrand (A2) does not enjoy any symmetry property as $\Lambda \rightarrow -\Lambda$.

Let's evaluate the following contour integral, making use of equation (A7) and (A8):

$$\begin{aligned} & \oint_{C_+} \frac{z dz}{\pi i(z^2 + k^2)} \frac{K_\nu(-izz)}{\tilde{f}z^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} \cdot (\text{y terms}) \\ &= \int_{R^+} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(-i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} \cdot (\text{y terms}) \\ &+ \int_{R_-} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(-i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} \cdot (\text{y terms}) \\ &+ \int_{C_+-R} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(-i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{-\nu\pi i/2} + e^{\nu\pi i/2}} \cdot (\text{y terms}) \\ &= I_+ + I_-, \end{aligned} \quad (\text{A9})$$

which gives rise precisely to the full integral (A1). Here R^+ , R^- and R denote the positive, negative, and the full real axis respectively. The choice of the contour C_+ is dictated by equation (A7) as the phase of Λ rotates smoothly from 0 to π . This contour integral can be evaluated, on the other hand, by picking up the residue at the pole $\Lambda = ke^{\pi i/2}$. Similarly, we can evaluate the contour integral along C_- :

$$\begin{aligned} & - \oint_{C_-} \frac{z dz}{\pi i(z^2 + k^2)} \frac{K_\nu(izz)}{\tilde{f}z^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \cdot (\text{y terms}) \\ &= - \int_{R^+} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \cdot (\text{y terms}) \\ &- \int_{R_-} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \cdot (\text{y terms}) \\ &- \int_{C_+-R} \frac{\Lambda d\Lambda}{\pi i(\Lambda^2 + k^2)} \frac{K_\nu(i\Lambda x)}{\tilde{f}\Lambda^{2\nu}e^{\nu\pi i/2} + e^{-\nu\pi i/2}} \cdot (\text{y terms}) \\ &= I_+ + I_- . \end{aligned} \quad (\text{A10})$$

This contour integral picks the residue at the pole $\Lambda = ke^{-\pi i/2}$ and leads to an identical result.

If $x < y$, both contour integrals given above diverge, in which case we should just exchange the role of x and y and carry out the above analysis similarly.

In summary, we find the propagator (A1) equals

$$I_+ + I_- = \frac{2K_\nu(kx) \left[I_\nu(ky) + \tilde{f}k^{2\nu} I_{-\nu}(ky) \right] \Theta(x-y)}{1 + \tilde{f}k^{2\nu}} + x \leftrightarrow y . \quad (\text{A11})$$

Appendix B: Some more details on the singularities of I_{ν, f_1, f_2}

We want to study the divergences of the integrals of the form [28]

$$I \equiv \int_0^{+\infty} z^{\frac{d}{2}} K_{\frac{d}{2}-1}(iz) J_\mu(\Lambda_1 z) J_\nu(\Lambda_2 z) dz$$

$$= \frac{\Gamma\left[1 + \frac{1}{2}(\mu + \nu)\right] \Gamma\left[\frac{1}{2}(d + \mu + \nu)\right] \Lambda_1^\mu \Lambda_2^\nu}{(i)^{d/2+1+\mu+\nu} \Gamma(\mu+1) \Gamma(\nu+1)} F_4\left[1 + \frac{1}{2}(\mu + \nu), \frac{1}{2}(d + \mu + \nu); \mu + 1, \nu + 1; \Lambda_1^2, \Lambda_2^2\right], \quad (\text{B1})$$

when $1 - |\Lambda_1| - |\Lambda_2| \rightarrow 0$. Within the region $|\Lambda_1| + |\Lambda_2| < 1$, the Appell hypergeometric function F_4 has a series expansion given by

$$F_4[a, b; c, d; \Lambda_1^2, \Lambda_2^2] = \sum_{m, n=0}^{\infty} \frac{(a)_m (b)_n}{m! n! (c)_m (d)_n} \Lambda_1^{2m} \Lambda_2^{2n}, \quad (\text{B2})$$

where the notation $(\cdot)_m$ is defined by

$$(a)_m \equiv \frac{\Gamma(a+m)}{\Gamma(a)}. \quad (\text{B3})$$

This expansion fails to converge whenever $1 - |\Lambda_1| - |\Lambda_2| \rightarrow 0$. One can, however, by choosing the integral contour of z carefully, extend the definition of I to the full (Λ_1, Λ_2) plane which only becomes singular when $1 \pm \Lambda_1 \pm \Lambda_2 \rightarrow 0$.

To isolate these singularities of I , we examine the asymptotic behavior of its integrand as $z \rightarrow \infty$ where the Bessel functions have an asymptotic representation given by

$$K_{\frac{d}{2}-1}(iz) \approx \sqrt{\frac{\pi}{2iz}} e^{-iz} \sum_{n=0}^{\infty} \frac{(\frac{d}{2}-1, n)}{(2iz)^n} \quad (\text{B4})$$

$$J_\mu(z) \approx \sqrt{\frac{1}{2\pi z}} \left[e^{i(z - \frac{\mu\pi}{2} - \frac{\pi}{4})} \sum_{n=0}^{\infty} \frac{i^n (\mu, n)}{(2z)^n} + e^{-i(z - \frac{\mu\pi}{2} - \frac{\pi}{4})} \sum_{n=0}^{\infty} \frac{(\mu, n)}{i^n (2z)^n} \right], \quad (-\pi < \arg z < \pi). \quad (\text{B5})$$

Here we defined notation

$$(\nu, n) \equiv \frac{\Gamma(\frac{1}{2} + \nu + n)}{n! \Gamma(\frac{1}{2} + \nu - n)}, \quad (\text{B6})$$

and it is useful to note that $(\nu, 0) = 1$ and $(\nu, n) = (-\nu, n)$. These expansions are only valid in the z -plane when its phase angle is between $-\pi$ and π . Using these formulae, we can easily derive an asymptotic expansion for the full integrand of I in the limit $z \rightarrow +\infty$ and formally evaluate the integral I by integrating the resultant expansion term by term:

$$I \sim \frac{1}{2^{\frac{d}{2}} \sqrt{\pi} \Lambda_1 \Lambda_2} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} \int_0^{\infty} dz \frac{(\frac{d}{2}-1, n-k-l) (\mu, k) (\nu, l)}{(2z)^{n+\frac{3-d}{2}} \Lambda_1^k \Lambda_2^l}$$

$$\cdot \left(e^{-i(\theta_\mu + \theta_\nu + \frac{3\pi}{4})} i^{2(k+l)-n} e^{-i[1-(\Lambda_1+\Lambda_2)]z} + e^{-i(\theta_\mu - \theta_\nu + \frac{\pi}{4})} i^{2k-n} e^{-i[1-(\Lambda_1-\Lambda_2)]z} \right.$$

$$\left. + e^{-i(\theta_\nu - \theta_\mu + \frac{\pi}{4})} i^{2l-n} e^{-i[1-(\Lambda_2-\Lambda_1)]x} + e^{i(\theta_\mu + \theta_\nu + \frac{\pi}{4})} i^{-n} e^{-i[1+(\Lambda_1+\Lambda_2)]x} \right)$$

$$\frac{1}{2^{\frac{3}{2}} \sqrt{\pi} \Lambda_1 \Lambda_2} \sum_{n=0}^{\infty} \sum_{k=0}^n \sum_{l=0}^{n-k} \frac{(\frac{d}{2}-1, n-k-l) (\mu, k) (\nu, l) \Gamma(\frac{d-1}{2} - n)}{2^n \Lambda_1^k \Lambda_2^l}$$

$$\cdot \left(e^{-i(\theta_\mu + \theta_\nu + \frac{(d-2)\pi}{4})} (-)^{k+l} [1 - (\Lambda_1 + \Lambda_2)]^{n+\frac{1-d}{2}} + e^{-i(\theta_\mu - \theta_\nu + \frac{d\pi}{4})} (-)^k [1 - (\Lambda_1 - \Lambda_2)]^{n+\frac{1-d}{2}} \right.$$

$$\left. + e^{-i(\theta_\nu - \theta_\mu + \frac{d\pi}{4})} (-)^l [1 - (\Lambda_2 - \Lambda_1)]^{n+\frac{1-d}{2}} + e^{i(\theta_\mu + \theta_\nu + \frac{d-2}{4}\pi)} [1 + (\Lambda_1 + \Lambda_2)]^{n+\frac{1-d}{2}} \right), \quad (\text{B7})$$

where we have defined the phase angle

$$\theta_\mu \equiv \frac{\mu\pi}{2}. \quad (\text{B8})$$

As explained in details in section III C, the “ \sim ” sign above must be understood as indicating that both sides equal apart from unknown finite functions. Therefore, terms that are not singular on the right hand side of the “ \sim ” sign are meaningless and should be discarded. In the domain we are interested in, $0 < \Lambda_{1,2} < 1$, and the poles of I are given by

$$I_{\nu,\infty}(i, \Lambda_1, \Lambda_2) \sim \frac{e^{-2i\theta_\nu} \Gamma\left(\frac{d-1}{2}\right)}{2i\sqrt{2\pi}\Lambda_1\Lambda_2} \left\{ \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{\frac{d-1}{2}}} + \left[\frac{d-1}{4} - \frac{(\nu, 1)}{d-3} \left(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2} \right) \right] \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{\frac{d-3}{2}}} \right\} \\ + O\left([1 - (\Lambda_1 + \Lambda_2)]^{-(d-5)/2}\right) + \text{finite things.} \quad (\text{B9})$$

More generally if $0 \leq \Lambda_{1,2} < +\infty$, I may contain other poles. Say, if $1 \pm (\Lambda_1 - \Lambda_2) \rightarrow 0$, we find in the same way that

$$I_{\nu,\infty}(i, \Lambda_1, \Lambda_2) \sim -\frac{\Gamma\left(\frac{d-1}{2}\right)}{2\sqrt{2\pi}\Lambda_1\Lambda_2} \left\{ \frac{1}{[1 - (\Lambda_1 - \Lambda_2)]^{\frac{d-1}{2}}} + \left[\frac{d-1}{4} - \frac{(\nu, 1)}{d-3} \left(\frac{1}{\Lambda_1} - \frac{1}{\Lambda_2} \right) \right] \frac{1}{[1 - (\Lambda_1 - \Lambda_2)]^{\frac{d-3}{2}}} \right\} \\ + O\left([1 - (\Lambda_1 + \Lambda_2)]^{-(d-5)/2}\right) + \Lambda_1 \leftrightarrow \Lambda_2 + \text{finite things.} \quad (\text{B10})$$

Since the asymptotic expansion (B4) and (B5) are not valid along the negative axis of z , this method can not be used to analyze the pole structures if either $\Lambda_{1,2}$ becomes negative. In particular, the pole of the form of $1/(1 + \Lambda_1 + \Lambda_2)^\alpha$ can not be properly derived as above, but they can be inferred easily by the symmetry properties of I when $\Lambda \rightarrow -\Lambda$.

If, instead of the standard Bessel functions $J_{\pm\nu}$, we wish to use $J_{\nu,f}$ as defined by (9) in the integral formulae for I as we should if the bulk scalar ϕ satisfies a mixed boundary condition, we must replace the asymptotic expansion of $J_{\pm\nu}$ by that of $J_{\nu,f}$. It's easily verified that

$$J_{\nu,f}(z) \stackrel{z \rightarrow \infty}{\approx} \sqrt{\frac{1}{2\pi z}} \left[e^{i(z - \theta_{\nu,f,\Lambda} - \frac{\pi}{4})} \sum_{n=0}^{\infty} \frac{i^n(\mu, n)}{(2z)^n} + e^{-i(z - \theta_{\nu,f,\Lambda} - \frac{\pi}{4})} \sum_{n=0}^{\infty} \frac{(\mu, n)}{i^n(2z)^n} \right] \quad (\text{B11})$$

which differs from those of $J_{\pm\nu}$ by merely a less trivial phase angle that we have denoted as $\theta_{\nu,f,\Lambda}$ above and is defined implicitly through the following equation:

$$\tan \theta_{\nu,f,\Lambda} \equiv \tan \frac{1 - \frac{\Lambda^{2\nu}}{f} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}}{1 + \frac{\Lambda^{2\nu}}{f} \frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}} \tan \theta_\nu. \quad (\text{B12})$$

Obviously $\theta_{\nu,f}$ interpolates between θ_ν and $\theta_{-\nu} = -\theta_\nu$ as f varies from ∞ to 0.

Therefore, it is straightforward to generalize the results given above and find, for example, when $1 - (\Lambda_1 + \Lambda_2) \rightarrow 0$:

$$I_{\nu,f_1,f_2}(i, \Lambda_1, \Lambda_2) \sim \frac{e^{-i(\theta_{\nu,f_1,\Lambda_1} + \theta_{\nu,f_2,\Lambda_2})} \Gamma\left(\frac{d-1}{2}\right)}{2i\sqrt{2\pi}\Lambda_1\Lambda_2} \left\{ \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{\frac{d-1}{2}}} + \left[\frac{d-1}{4} - \frac{(\nu, 1)}{d-3} \left(\frac{1}{\Lambda_1} + \frac{1}{\Lambda_2} \right) \right] \frac{1}{[1 - (\Lambda_1 + \Lambda_2)]^{\frac{d-3}{2}}} \right\} \\ + O\left([1 - (\Lambda_1 + \Lambda_2)]^{-(d-5)/2}\right) + \text{finite things.} \quad (\text{B13})$$

which is essentially identical to that of $I_{\nu,\infty}$ except that $\theta_{\nu,f,\Lambda}$ depends both on f and Λ now [38]. This is, of course, necessary if any nontrivial results are to be expected.

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- [1] J. M. Maldacena, Adv. Theor. Math. Phys. **2**, 231 (1998), arXiv:hep-th/9711200.
 - [2] S. S. Gubser, I. R. Klebanov, and A. M. Polyakov, Phys. Lett. **B428**, 105 (1998), arXiv:hep-th/9802109.
 - [3] E. Witten, Adv. Theor. Math. Phys. **2**, 253 (1998), arXiv:hep-th/9802150.
 - [4] S. A. Hartnoll, Class. Quant. Grav. **26**, 224002 (2009), arXiv:0903.3246 [hep-th].
 - [5] F. Denef, S. A. Hartnoll, and S. Sachdev, Class. Quant. Grav. **27**, 125001 (2010), arXiv:0908.2657 [hep-th].
 - [6] F. Denef, S. A. Hartnoll, and S. Sachdev, Phys. Rev. **D80**, 126016 (2009), arXiv:0908.1788 [hep-th].
 - [7] S. Caron-Huot and O. Saremi(2009), arXiv:0909.4525 [hep-th].
 - [8] D. Anninos, S. A. Hartnoll, and N. Iqbal(2010), arXiv:1005.1973 [hep-th].

- [9] D. F. Mross, J. McGreevy, H. Liu, and T. Senthil(2010), arXiv:1003.0894 [cond-mat.str-el].
- [10] M. Fujita, Y. Hikida, S. Ryu, and T. Takayanagi, JHEP **12**, 065 (2008), arXiv:0810.5394 [hep-th].
- [11] E. Witten(2001), arXiv:hep-th/0112258.
- [12] O. Aharony, M. Berkooz, and E. Silverstein, JHEP **08**, 006 (2001), arXiv:hep-th/0105309.
- [13] M. Berkooz, A. Sever, and A. Shomer, JHEP **05**, 034 (2002), arXiv:hep-th/0112264.
- [14] O. Aharony, M. Berkooz, and B. Katz, JHEP **10**, 097 (2005), arXiv:hep-th/0504177.
- [15] I. Papadimitriou, JHEP **05**, 075 (2007), arXiv:hep-th/0703152.
- [16] T. Hartman and L. Rastelli, JHEP **01**, 019 (2008), arXiv:hep-th/0602106.
- [17] S. Elitzur, A. Giveon, M. Porrati, and E. Rabinovici, JHEP **02**, 006 (2006), arXiv:hep-th/0511061.
- [18] One might wonder why the effect of disorder is so weak. This is actually related to the fact that in the replica trick implemented in the bulk we have implicitly assumed that the strength of the coupling to the random disorder is only of order $1/N^2$. See for example the discussion in [12].
- [19] This limit, however is quite different from the usual infrared divergence in the loop correction of the photon propagator, which appears as the external invariant d -momentum goes on-shell i.e. $p_\mu p^\mu \rightarrow 0$. The singularity here occurs even for finite external invariant d -momentum.
- [20] I. R. Klebanov and E. Witten, Nucl. Phys. **B556**, 89 (1999), arXiv:hep-th/9905104.
- [21] S. S. Gubser and I. Mitra, Phys. Rev. **D67**, 064018 (2003), arXiv:hep-th/0210093.
- [22] Another straightforward way of obtaining this propagator is expanding all the fields in terms of the properly chosen radial wave-functions and express the action in terms of the expansion coefficients. The inverse of the resultant kinetic term in the action leads immediately to equation (8).
- [23] G. Chalmers and K. Schalm, Nucl. Phys. **B554**, 215 (1999), arXiv:hep-th/9810051.
- [24] E. D'Hoker, D. Z. Freedman, S. D. Mathur, A. Matusis, and L. Rastelli, Nucl. Phys. **B562**, 330 (1999), arXiv:hep-th/9902042.
- [25] H. Liu and A. A. Tseytlin, Nucl. Phys. **B533**, 88 (1998), arXiv:hep-th/9804083.
- [26] H. Liu and A. A. Tseytlin, Phys. Rev. **D59**, 086002 (1999), arXiv:hep-th/9807097.
- [27] For a detailed discussion see for example [37, 39, 40].
- [28] W. N. Bailey, Proc. London Math. Soc **s2-40**, 37 (1936).
- [29] Take $d = 4$ and the leading term in the asymptotic expansion as an example, the relevant integrals is

$$I_{\text{leading}} = \text{finite piece} + \int_L^\infty dz \sqrt{z} e^{-i\tilde{p}z} = \frac{e^{-iL\tilde{p}}\sqrt{L}}{i\tilde{p}} + \frac{\sqrt{\pi}\text{erf}(\sqrt{iL\tilde{p}})}{(2i\tilde{p})^{3/2}}, \quad (14)$$
 where \tilde{p} stands for any one of the four combinations $p \pm \Lambda_1 \pm \Lambda_2$. To identify the poles of \tilde{p} , one takes the limit $\tilde{p} \rightarrow 0$ and finds that the leading divergence is precisely given by $\sqrt{\pi}/(2i\tilde{p})^{3/2}$. Anything that depends on L remains finite in the limit $\tilde{p} \rightarrow 0$. The apparent additional $1/\tilde{p}$ term in the above equation is cancelled out at this order by an identical piece that arises in the expansion of the error function near the pole.
- [30] Although one needs to extend the definition for I_{ν, f_1, f_2} outside $0 \leq \Lambda_1 + \Lambda_2 \leq 1$ by choosing the integral contour for z with special care.
- [31] L.-Y. Hung and Y. Shang, work in progress.
- [32] N. Iqbal and H. Liu, Fortsch. Phys. **57**, 367 (2009), arXiv:0903.2596 [hep-th].
- [33] Cardy, J., ArXiv Condensed Matter e-prints(nov 2001), arXiv:cond-mat/0111031.
- [34] S. A. Hartnoll and C. P. Herzog, Phys. Rev. **D77**, 106009 (2008), arXiv:0801.1693 [hep-th].
- [35] O. Aharony, A. B. Clark, and A. Karch, Phys. Rev. **D74**, 086006 (2006), arXiv:hep-th/0608089.
- [36] E. Kiritsis, JHEP **11**, 049 (2006), arXiv:hep-th/0608088.
- [37] K. Skenderis and B. C. van Rees, JHEP **05**, 085 (2009), arXiv:0812.2909 [hep-th].
- [38] In this case one can again set $p = 1$ which amounts to a scaling of f . Rigorously speaking, $\theta_{\nu, f, \Lambda}$ depends on both f and p , but only through their ratio f/p .
- [39] D. T. Son and A. O. Starinets, JHEP **09**, 042 (2002), arXiv:hep-th/0205051.
- [40] C. P. Herzog and D. T. Son, JHEP **03**, 046 (2003), arXiv:hep-th/0212072.